Mathematics of Information — Cheatsheet

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1 A Short Course on Frame Theory

1.1 Examples of Signal Expansions

Orthonormal Basis (ONB) in $\mathbb{R}^2$: Consider $\mathbf{e}_1 = [1, 0]^\top, \mathbf{e}_2 = [0, 1]^\top$. Every signal $\mathbf{x} \in \mathbb{R}^2$ expands as

$$\mathbf{x} = \langle \mathbf{x}, \mathbf{e}_1 \rangle \mathbf{e}_1 + \langle \mathbf{x}, \mathbf{e}_2 \rangle \mathbf{e}_2.$$

Analysis and Synthesis Matrices

Define expansion coefficients in vector form: $\mathbf{c} = \mathbf{T}\mathbf{x}$ where

$$\mathbf{T} = \begin{bmatrix} \mathbf{e}_1^\top \\ \mathbf{e}_2^\top \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is the *analysis matrix* (produces coefficients). Reconstruct signal from coefficients via

$$\mathbf{x} = \mathbf{T}^\top \mathbf{c} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{bmatrix} \mathbf{c}$$

where $\mathbf{T}^\top$ is the *synthesis matrix*. For ONB: perfect reconstruction holds $\mathbf{x} = \mathbf{T}^\top \mathbf{T}\mathbf{x}$.

Biorthonormal Bases in $\mathbb{R}^2$: Take non-orthogonal vectors $\mathbf{e}_1 = [1, 0]^\top, \mathbf{e}_2 = \frac{1}{\sqrt{2}}[1, 1]^\top$. Analysis matrix $\mathbf{T} = \begin{bmatrix} 1 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ is invertible. To reconstruct

$$\mathbf{x} = \langle \mathbf{x}, \mathbf{e}_1 \rangle \tilde{\mathbf{e}}_1 + \langle \mathbf{x}, \mathbf{e}_2 \rangle \tilde{\mathbf{e}}_2$$

we set $[\tilde{\mathbf{e}}_1 \, \tilde{\mathbf{e}}_2] = \mathbf{T}^{-1}$, giving $\tilde{\mathbf{e}}_1 = [1, -1]^\top, \tilde{\mathbf{e}}_2 = [0, \sqrt{2}]^\top$.

Biorthonormality Property

The two bases satisfy $\langle \mathbf{e}_j, \tilde{\mathbf{e}}_k \rangle = \delta_{jk}$ (Kronecker delta). Matrix form: $\mathbf{T}[\tilde{\mathbf{e}}_1 \, \tilde{\mathbf{e}}_2] = \mathbf{I}_2$, enabling perfect reconstruction $\mathbf{x} = \tilde{\mathbf{T}}^\top \mathbf{T}\mathbf{x}$ where $\tilde{\mathbf{T}}^\top = [\tilde{\mathbf{e}}_1 \, \tilde{\mathbf{e}}_2]$.

Overcomplete Expansion (Frame) in $\mathbb{R}^2$: Three vectors in 2D are linearly dependent. Consider $\mathbf{g}_1 = [1, 0]^\top, \mathbf{g}_2 = [0, 1]^\top, \mathbf{g}_3 = [1, -1]^\top$ with $\mathbf{g}_3 = \mathbf{g}_1 - \mathbf{g}_2$. Analysis matrix $\mathbf{T} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}$ is 3×2 (tall, rank-deficient inverse).

We can represent $\mathbf{x}$ using these three vectors in multiple ways:

$$\mathbf{x} = \langle \mathbf{x}, \mathbf{g}_1 \rangle \tilde{\mathbf{g}}_1 + \langle \mathbf{x}, \mathbf{g}_2 \rangle \tilde{\mathbf{g}}_2 + \langle \mathbf{x}, \mathbf{g}_3 \rangle \tilde{\mathbf{g}}_3$$

Design Freedom in Frames

Two valid dual frames exist:

1. $\tilde{\mathbf{g}}_1 = \mathbf{g}_1, \tilde{\mathbf{g}}_2 = \mathbf{g}_2, \tilde{\mathbf{g}}_3 = \mathbf{0}$ (ignores third vector)
2. $\tilde{\mathbf{g}}_1 = 2\mathbf{g}_1, \tilde{\mathbf{g}}_2 = \mathbf{g}_2 - \mathbf{g}_1, \tilde{\mathbf{g}}_3 = -\mathbf{g}_1$ (exploits redundancy)

Since $\mathbf{T}$ has infinitely many left-inverses $\tilde{\mathbf{T}}^\top$ satisfying $\tilde{\mathbf{T}}^\top \mathbf{T} = \mathbf{I}_2$, we have infinitely many dual frames. This *redundancy provides design freedom* in choosing how to synthesize the signal.

1.2 Signal Expansions in Finite-Dimensional Hilbert Spaces

1.2.1 Orthonormal Bases

Definition 1.4: Orthonormal Basis (ONB)

A set of vectors $\{\mathbf{e}_k\}_{k=1}^M$, $\mathbf{e}_k \in \mathbb{C}^M$, is an ONB for $\mathbb{C}^M$ if:

1. $\operatorname{span}\{\mathbf{e}_k\}_{k=1}^M = \mathbb{C}^M$ (spanning)
2. $\langle \mathbf{e}_k, \mathbf{e}_j \rangle = \delta_{kj}$ (orthonormality)

Why ONBs matter: Every signal in $\mathbb{C}^M$ decomposes uniquely. For any $\mathbf{x} \in \mathbb{C}^M$:

$$\mathbf{x} = \sum_{k=1}^{M} \langle \mathbf{x}, \mathbf{e}_k \rangle \mathbf{e}_k$$

The coefficients $c_k = \langle \mathbf{x}, \mathbf{e}_k \rangle$ are found via orthogonal projection—no linear system solving needed.

Example: Standard basis $\mathbf{e}_1 = [1, 0]^\top, \mathbf{e}_2 = [0, 1]^\top$ in $\mathbb{R}^2$ is an ONB. For $\mathbf{x} = [3, 4]^\top$:

$$c_1 = \langle \mathbf{x}, \mathbf{e}_1 \rangle = 3, \quad c_2 = \langle \mathbf{x}, \mathbf{e}_2 \rangle = 4.$$

So $\mathbf{x} = 3\mathbf{e}_1 + 4\mathbf{e}_2$.

Analysis and Synthesis with ONBs

Stack basis vectors into analysis matrix:

$$\mathbf{T} = \begin{bmatrix} \mathbf{e}_1^{\mathrm{H}} \\ \vdots \\ \mathbf{e}_M^{\mathrm{H}} \end{bmatrix}$$

Coefficients: $\mathbf{c} = \mathbf{T}\mathbf{x}$. Since $\mathbf{e}_k$ are orthonormal, $\mathbf{T}$ is unitary: $\mathbf{T}^{\mathrm{H}}\mathbf{T} = \mathbf{I}_M$.
Synthesis: reconstruct from $\mathbf{c}$ via

$$\mathbf{x} = \mathbf{T}^{\mathrm{H}} \mathbf{c}$$

where $\mathbf{T}^{\mathrm{H}} = [\mathbf{e}_1 \cdots \mathbf{e}_M]$ is the synthesis matrix.

Parseval's Theorem (Energy Conservation)

For ONBs:

$$\|\mathbf{x}\|^2 = \sum_{k=1}^{M} |c_k|^2 = \|\mathbf{c}\|^2$$

The norm of the signal equals the norm of its coefficient vector. This holds because $\mathbf{T}$ is unitary.

Why this works: Unitarity means $\mathbf{T}^{\mathrm{H}}\mathbf{T} = \mathbf{I}_M$, so

$$\|\mathbf{c}\|^2 = \mathbf{c}^{\mathrm{H}}\mathbf{c} = \mathbf{x}^{\mathrm{H}}\mathbf{T}^{\mathrm{H}}\mathbf{T}\mathbf{x} = \mathbf{x}^{\mathrm{H}}\mathbf{x} = \|\mathbf{x}\|^2.$$

Key Property: Perfect Reconstruction

From $\mathbf{c} = \mathbf{T}\mathbf{x}$, we recover $\mathbf{x}$ exactly:

$$\mathbf{T}^{\mathrm{H}}\mathbf{c} = \mathbf{T}^{\mathrm{H}}\mathbf{T}\mathbf{x} = \mathbf{x}.$$

No information loss in analysis or synthesis.

1.2.2 General (Biorthonormal) Bases

We relax orthonormality and allow non-orthogonal bases. Any basis $\{\mathbf{e}_k\}_{k=1}^M$ (spanning, linearly independent) has a **unique dual basis** $\{\tilde{\mathbf{e}}_k\}_{k=1}^M$ satisfying:

$$\langle \mathbf{e}_k, \tilde{\mathbf{e}}_j \rangle = \delta_{kj}$$

(biorthonormality). Perfect reconstruction holds:

$$\mathbf{x} = \sum_{k=1}^{M} \langle \mathbf{x}, \mathbf{e}_k \rangle \tilde{\mathbf{e}}_k$$

In matrix form: $\tilde{\mathbf{T}}^{\mathrm{H}}\mathbf{T} = \mathbf{I}_M$, where $\mathbf{T}$ is the analysis matrix of $\{\mathbf{e}_k\}$ and $\tilde{\mathbf{T}}^{\mathrm{H}}$ is the synthesis matrix of $\{\tilde{\mathbf{e}}_k\}$. Since $\mathbf{T}$ is square and full-rank, $\tilde{\mathbf{T}}^{\mathrm{H}} = \mathbf{T}^{-1}$ is unique.

Key distinction from ONBs: Biorthonormal bases are **not** norm-preserving. The norm of coefficients obeys:

$$\lambda_{\min}(\mathbf{T}^{\mathrm{H}}\mathbf{T})\|\mathbf{x}\|^2 \leq \|\mathbf{c}\|^2 \leq \lambda_{\max}(\mathbf{T}^{\mathrm{H}}\mathbf{T})\|\mathbf{x}\|^2$$

1.2.3 Redundant Signal Expansions (Frames in Finite Dimensions)

Consider $N > M$ vectors $\{\mathbf{g}_1, \ldots, \mathbf{g}_N\}$, $\mathbf{g}_k \in \mathbb{C}^M$, that **span** $\mathbb{C}^M$ but are linearly dependent. The analysis matrix $\mathbf{T} \in \mathbb{C}^{N \times M}$ (tall) has full column rank.

For any signal $\mathbf{x} \in \mathbb{C}^M$, compute coefficients:

$$\mathbf{c} = \mathbf{T}\mathbf{x} \in \mathbb{C}^N$$

A set $\{\tilde{\mathbf{g}}_1, \ldots, \tilde{\mathbf{g}}_N\}$ is a **dual frame** if:

$$\mathbf{x} = \sum_{k=1}^{N} \langle \mathbf{x}, \mathbf{g}_k \rangle \tilde{\mathbf{g}}_k \quad \text{for all } \mathbf{x} \in \mathbb{C}^M$$

This requires $\tilde{\mathbf{T}}^{\mathrm{H}}\mathbf{T} = \mathbf{I}_M$, where $\tilde{\mathbf{T}}^{\mathrm{H}} \in \mathbb{C}^{M \times N}$ is a synthesis matrix.

Theorem 1.6 (Moore-Penrose Inverse)

Let $\mathbf{T} \in \mathbb{C}^{N \times M}$ with $N \geq M$ and $\operatorname{rank}(\mathbf{T}) = M$. The Moore-Penrose inverse is:

$$\mathbf{T}^{\dagger} = (\mathbf{T}^{\mathrm{H}}\mathbf{T})^{-1}\mathbf{T}^{\mathrm{H}}$$

This is a left-inverse: $\mathbf{T}^{\dagger}\mathbf{T} = \mathbf{I}_M$.
All left-inverses (dual frame syntheses) are parametrized as:

$$\mathbf{L} = \mathbf{T}^{\dagger} + \mathbf{M}(\mathbf{I}_N - \mathbf{T}\mathbf{T}^{\dagger})$$

for arbitrary $\mathbf{M} \in \mathbb{C}^{M \times N}$. This gives **infinitely many dual frames** when $N > M$ (design freedom).

1.3 Frames for General Hilbert Spaces

Motivation: From Finite to Infinite Dimensions

Finite-dimensional analysis matrices $\mathbf{T}$ generalize to **analysis operators** $\mathbb{T} : \mathcal{H} \to \ell^2$ in infinite-dimensional Hilbert spaces. A frame formalizes the concept of redundant signal expansions.

What is a Bessel sequence? A set $\{g_k\}_{k \in \mathcal{K}}$ is a Bessel sequence if:

$$\sum_{k \in \mathcal{K}} |\langle x, g_k \rangle|^2 < \infty \quad \text{for all } x \in \mathcal{H}$$

This ensures the analysis operator $\mathbb{T} : x \mapsto \{\langle x, g_k \rangle\}_{k \in \mathcal{K}}$ maps $\mathcal{H}$ into ℓ^2.

Definition 1.8: Frame

A Bessel sequence $\{g_k\}_{k \in \mathcal{K}}, g_k \in \mathcal{H}$, is a **frame** for $\mathcal{H}$ if there exist constants $0 < A \leq B < \infty$ (frame bounds) such that:

$$A\|x\|^2 \leq \sum_{k \in \mathcal{K}} |\langle x, g_k \rangle|^2 \leq B\|x\|^2 \quad \text{for all } x \in \mathcal{H}$$

Equivalently: $A\|x\|^2 \leq \|\mathbb{T}x\|^2 \leq B\|x\|^2$.

Key properties of frames:

- **Completeness**: If $A > 0$ and $\langle x, g_k \rangle = 0$ for all k, then $x = 0$
- **Boundedness**: If $B < \infty$, then $\mathbb{T}$ is continuous
- **Lower bound → left-invertibility**: $A > 0$ ensures perfect reconstruction is possible

Example 1.9: Redundant ONB If $\{e_k\}$ is an ONB and $\{g_k\} = \{e_1, e_1, e_2, e_2, \ldots\}$ (each element repeated), then:

$$\sum_{k=1}^{\infty} |\langle x, g_k \rangle|^2 = 2\|x\|^2$$

So $\{g_k\}$ is a frame with bounds $A = B = 2$.

Example 1.10: Weighted frame Starting from ONB $\{e_k\}$, construct $\{g_k\} = \{e_1, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{3}}e_3, \ldots\}$ where e_k appears k times with weight $\frac{1}{\sqrt{k}}$. This gives a tight frame with $A = B = 1$.

The adjoint operator: For a bounded linear operator $\mathbb{T} : \mathcal{H} \to \ell^2$, define the **adjoint** $\mathbb{T}^* : \ell^2 \to \mathcal{H}$ by:

$$\langle \mathbb{T}x, c \rangle = \langle x, \mathbb{T}^* c \rangle$$

For frame analysis, $\mathbb{T}^*$ acts as:

$$\mathbb{T}^*\{c_k\}_{k \in \mathcal{K}} = \sum_{k \in \mathcal{K}} c_k g_k$$

This generalizes the Hermitian transpose $\mathbf{T}^{\mathrm{H}}$ from finite to infinite dimensions.

Synthesis Operator $\mathbb{T}^*$: Adjoint vs. Dual Synthesis

The **synthesis operator** $\mathbb{T}^* : l^2 \to \mathcal{H}$ is the adjoint of the analysis operator: $\mathbb{T}^* c = \sum c_k g_k$.
Critical distinction: $\mathbb{T}^* \neq \mathbb{T}^*$ (dual synthesis):

- $\mathbb{T}^*$: Adjoint only. $\mathbb{T}^*\mathbb{T}x = \mathbb{S}x \neq x$ (NO perfect reconstruction).
- $\tilde{\mathbb{T}}^* = \mathbb{S}^{-1}\mathbb{T}^*$: Gives $\tilde{\mathbb{T}}^*\mathbb{T}x = x$ (perfect reconstruction).

Why? In redundant frames, $\mathbb{T}^*\mathbb{T} = \mathbb{S} \neq I$. Reconstruction requires:

$$x = \mathbb{T}^*\mathbb{S}^{-1}\mathbb{T}x \quad \text{or equivalently} \quad x = \tilde{\mathbb{T}}^*\mathbb{T}x$$

Example: Why $\mathbb{T}^*$ alone fails
Using $g_1 = g_2 = (1,0)^T$, $g_3 = (0,1)^T$: For $x = (a,b)^T$, coefficients are $c = \mathbb{T}x = (a,a,b)^T$.
Then $\mathbb{T}^*c = (a+a,b)^T = (2a,b)^T \neq x$. Apply $\mathbb{S}^{-1} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}$ to recover: $\mathbb{S}^{-1}(2a,b)^T = (a,b)^T$ ✓

1.3.1 The Frame Operator

Definition 1.14: Frame Operator

For a frame $\{g_k\}_{k \in \mathcal{K}}$ with analysis operator $\mathbb{T}$, the **frame operator** is:

$$\mathbb{S} = \mathbb{T}^*\mathbb{T} : \mathcal{H} \to \mathcal{H}$$

Acting on $x \in \mathcal{H}$:

$$\mathbb{S}x = \sum_{k \in \mathcal{K}} \langle x, g_k \rangle g_k$$

Why is $\mathbb{S}$ central?

- The frame condition $A\|x\|^2 \le \langle \mathbb{S}x, x\rangle \le B\|x\|^2$ directly characterizes frames via $\mathbb{S}$
- Perfect reconstruction: $x = \mathbb{T}^*\mathbb{S}^{-1}\mathbb{T}x$ (requires $\mathbb{S}$ invertible)
- In finite dimensions: $\mathbb{S} = \mathbf{T}^{\mathrm{H}}\mathbf{T}$ (Gram matrix of frame elements)

Theorem 1.15: Properties of the Frame Operator

1. **Linear and bounded**: $\mathbb{S}$ is continuous (composition of bounded operators)
2. **Self-adjoint**: $\mathbb{S}^* = (\mathbb{T}^*\mathbb{T})^* = \mathbb{T}^*\mathbb{T} = \mathbb{S}$
3. **Positive definite**: $\langle \mathbb{S}x, x\rangle = \|\mathbb{T}x\|^2 \ge A\|x\|^2 > 0$ for all $x \ne 0$
4. **Has unique positive square root**: There exists a unique self-adjoint positive operator $\mathbb{S}^{1/2}$ with $\mathbb{S} = \mathbb{S}^{1/2}\mathbb{S}^{1/2}$

Consequence: $\mathbb{S}$ is invertible, and the inverse $\mathbb{S}^{-1}$ is also self-adjoint, positive, bounded.

Proof outline:

- (1) Composition of bounded operators
- (2) $(AB)^* = B^*A^*$ gives self-adjointness
- (3) Lower bound: $\langle \mathbb{S}x, x\rangle \ge A\|x\|^2 > 0$
- (4) Spectral theorem: positive operators have unique positive square roots

Proof details:

- Property 1 follows because $\mathbb{S}$ is the composition $\mathbb{T}^*\mathbb{T}$ of bounded operators.
- Property 2 uses $(AB)^* = B^*A^*$, so $(\mathbb{T}^*\mathbb{T})^* = \mathbb{T}^*(\mathbb{T}^*)^* = \mathbb{T}^*\mathbb{T}$.
- Property 3: Lower frame bound gives $\langle \mathbb{S}x, x\rangle = \sum |\langle x, g_k\rangle|^2 \ge A\|x\|^2 > 0$ for $x \ne 0$.
- Property 4 follows from general functional analysis: every self-adjoint positive bounded operator has unique self-adjoint positive square root (by spectral theorem).

Theorem 1.17: Frame Bounds from Spectrum

The **optimal** frame bounds are the extremal spectral values of $\mathbb{S}$:

$$A = \lambda_{\min}(\mathbb{S}) \quad \text{and} \quad B = \lambda_{\max}(\mathbb{S})$$

Intuition: Frame bounds characterize how $\mathbb{S}$ stretches/shrinks vectors. Spectral bounds are tight (optimal).

Numerical Example: Frame Operator in $\mathbb{R}^2$

Consider redundant frame in $\mathbb{R}^2$:

$$g_1 = \begin{pmatrix}1\\0\end{pmatrix}, \quad g_2 = \begin{pmatrix}1\\0\end{pmatrix}, \quad g_3 = \begin{pmatrix}0\\1\end{pmatrix}$$

(First element repeated twice.) The analysis operator is $\mathbb{T}x = \{\langle x, g_k\rangle\}_{k=1}^3$, represented as:

$$\mathbb{T} = \begin{pmatrix}1 & 0\\1 & 0\\0 & 1\end{pmatrix}$$

The Gram matrix is:

$$\mathbb{S} = \mathbb{T}^*\mathbb{T} = \begin{pmatrix}1 & 1 & 0\\1 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\1 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}2 & 0\\0 & 1\end{pmatrix}$$

Eigenvalues: $\lambda_{\min} = 1$, $\lambda_{\max} = 2$, so frame bounds are $A = 1, B = 2$.

Direct verification: For $x = \begin{pmatrix}a\\b\end{pmatrix}$,

$$\sum_{k=1}^{3} |\langle x, g_k\rangle|^2 = a^2 + a^2 + b^2 = 2a^2 + b^2$$

With $\|x\|^2 = a^2 + b^2$, we confirm:

$$(a^2 + b^2) \le 2a^2 + b^2 \le 2(a^2 + b^2) \quad \checkmark$$

1.3.2 The Canonical Dual Frame

Theorem 1.19: Canonical Dual Frame

Let $\{g_k\}_{k\in\mathcal{K}}$ be a frame with frame operator $\mathbb{S} = \mathbb{T}^*\mathbb{T}$ and bounds A, B. The **canonical dual frame** is:

$$\tilde{g}_k = \mathbb{S}^{-1}g_k, \quad k \in \mathcal{K}$$

This is also a frame for $\mathcal{H}$ with bounds $\tilde{A} = 1/B$ and $\tilde{B} = 1/A$.

Why is the canonical dual frame important?

- **Perfect reconstruction**: $x = \sum_{k\in\mathcal{K}} \langle x, g_k\rangle \tilde{g}_k$ for all $x \in \mathcal{H}$
- **Uniqueness**: The dual frame is the unique frame for perfect reconstruction
- **Reciprocal duality**: If $\{\tilde{g}_k\}$ is the dual of $\{g_k\}$, then $\{g_k\}$ is the dual of $\{\tilde{g}_k\}$

Key relationship: The analysis operator of the dual frame is:

$$\tilde{\mathbb{T}} = \mathbb{T}\mathbb{S}^{-1} = \mathbb{T}(\mathbb{T}^*\mathbb{T})^{-1}$$

and its frame operator is $\tilde{\mathbb{S}} = \mathbb{S}^{-1}$.

Proof outline:

- Invertibility: $A > 0$ ensures $\mathbb{S}$ is invertible by spectral theorem
- Dual bounds: Spectral analysis gives $\tilde{A} = 1/B$, $\tilde{B} = 1/A$
- Analysis operator: $\tilde{\mathbb{T}}x = \mathbb{T}(\mathbb{S}^{-1}x)$ (change of variables using self-adjointness)

Proof details:

- **Frame operator invertibility**: Positive lower bound $A > 0$ ensures $\mathbb{S}$ is invertible (spectral theorem)
- **Bounds of dual**: Using $\langle \mathbb{S}^{-1}x, x\rangle = \langle \mathbb{S}(\mathbb{S}^{-1}x), \mathbb{S}^{-1}x\rangle$ and spectral properties gives $1/B \le \frac{\langle \mathbb{S}^{-1}x, x\rangle}{\|x\|^2} \le 1/A$
- **Analysis operator**: $\tilde{\mathbb{T}}x = \{\langle x, \tilde{g}_k\rangle\}_{k\in\mathcal{K}} = \{\langle x, \mathbb{S}^{-1}g_k\rangle\}_{k\in\mathcal{K}} = \{\langle \mathbb{S}^{-1}x, g_k\rangle\}_{k\in\mathcal{K}} = \mathbb{T}(\mathbb{S}^{-1}x)$

Numerical Example: Dual Frame in $\mathbb{R}^2$

Continuing the previous example with $g_1 = g_2 = (1,0)^T$ and $g_3 = (0,1)^T$ where $\mathbb{S} = \begin{pmatrix}2 & 0\\0 & 1\end{pmatrix}$:

$$\mathbb{S}^{-1} = \begin{pmatrix}1/2 & 0\\0 & 1\end{pmatrix}$$

The dual frame elements are:

$$\tilde{g}_1 = \mathbb{S}^{-1}g_1 = \begin{pmatrix}1/2\\0\end{pmatrix}, \quad \tilde{g}_2 = \begin{pmatrix}1/2\\0\end{pmatrix}, \quad \tilde{g}_3 = \begin{pmatrix}0\\1\end{pmatrix}$$

Verify reconstruction: For $x = (a,b)^T$:

$$\sum_{k=1}^{3} \langle x, g_k\rangle \tilde{g}_k = a\cdot\frac{1}{2}(1,0)^T + a\cdot\frac{1}{2}(1,0)^T + b\cdot(0,1)^T = (a,b)^T = x$$

The dual frame has bounds $\tilde{A} = 1/B = 1/2$ and $\tilde{B} = 1/A = 1$.

1.3.3 Signal Expansions (Reconstruction)

Theorem 1.22: Frame Reconstruction

Let $\{g_k\}$ and $\{\tilde{g}_k\}$ be canonical dual frames. Every signal $x \in \mathcal{H}$ has two perfect reconstructions:

$$x = \sum_{k\in\mathcal{K}} \langle x, \tilde{g}_k\rangle g_k = \sum_{k\in\mathcal{K}} \langle x, g_k\rangle \tilde{g}_k$$

Equivalently: $\mathbb{T}^*\tilde{\mathbb{T}} = \tilde{\mathbb{T}}^*\mathbb{T} = \mathbb{I}_{\mathcal{H}}$.

In operator form:

$$x = \mathbb{T}^*\mathbb{S}^{-1}\mathbb{T}x = \mathbb{S}^{-1}\mathbb{S}x = x \quad \checkmark$$

Why two forms? The dual frame provides a natural way to recover x directly from the frame coefficients $\{\langle x, g_k\rangle\}$, without explicitly computing $\mathbb{S}^{-1}$. Both are mathematically equivalent.

1.3.4 Tight Frames

Definition 1.25: Tight Frame

A frame $\{g_k\}_{k\in\mathcal{K}}$ with **equal** tight bounds $A = B$ is called **tight**.

Theorem 1.26: Tight Frame Characterization

A frame is tight with bound A if and only if:

$$\mathbb{S} = A\mathbb{I}_{\mathcal{H}} \quad \Longleftrightarrow \quad x = \frac{1}{A}\sum_{k\in\mathcal{K}} \langle x, g_k\rangle g_k$$

This is **Parseval's identity** for tight frames. The dual is simple: $\tilde{g}_k = \frac{1}{A}g_k$.

Why tight frames matter:

- Perfect reconstruction without inverting $\mathbb{S}$ — just scale by $1/A$
- Example: Mercedes-Benz frame in $\mathbb{R}^2$: three unit vectors at 120 angles form tight frame with $A = 3/2$

Theorem 1.28: Normalize Any Frame to Tight
Given any frame $\{g_k\}$ with frame operator $\mathbb{S}$, the scaled frame $\{\mathbb{S}^{-1/2}g_k\}$ is tight with bound $A = 1$:

$$x = \sum_{k\in\mathcal{K}} \langle x, \mathbb{S}^{-1/2}g_k\rangle \mathbb{S}^{-1/2}g_k$$

This provides a systematic way to design tight frames from redundant ones.

1.3.5 Exact Frames

Definition 1.32: Exact Frame

A frame $\{g_k\}_{k\in\mathcal{K}}$ is **exact** if removing any single element leaves an incomplete set. Otherwise it is **inexact** (redundant).

Key equivalences:

- Exact frame = **Riesz basis** = frame with no redundancy
- Analysis operator $\mathbb{T}$ is **surjective** (onto ℓ^2)
- Representation in frame is **unique**: if $x = \sum c_k g_k = \sum a_k g_k$, then $c_k = a_k$ for all k (for exact frames, coefficients are $\langle x, \tilde{g}_k\rangle$ with minimal ℓ^2-norm)

Biorthonormality: Exact frames and their canonical duals satisfy:

$$\langle g_k, \tilde{g}_m\rangle = \delta_{k,m}$$

This is the infinite-dimensional generalization of biorthonormal bases.

Why exactness matters:

- Frames are generalizations of bases: exact = no redundancy, inexact = has redundancy
- Reconstruction coefficients in exact frames have **unique minimum ℓ^2-norm** (canonical dual gives this)
- Tight exact frames = orthonormal bases (special case: $A = 1$, no redundancy)