Mathematics of Information — Cheatsheet

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1.3 Frames for General Hilbert Spaces

Motivation: From Finite to Infinite Dimensions

Finite-dimensional analysis matrices $\mathbf{T}$ generalize to **analysis operators** $\mathbb{T} : \mathcal{H} \to \ell^2$ in infinite-dimensional Hilbert spaces. A frame formalizes the concept of redundant signal expansions.

What is a Bessel sequence? A set $\{g_k\}_{k\in\mathcal{K}}$ is a Bessel sequence if:

$$\sum_{k\in\mathcal{K}} |\langle x, g_k\rangle|^2 < \infty \quad \text{for all } x \in \mathcal{H}$$

This ensures the analysis operator $\mathbb{T} : x \mapsto \{\langle x, g_k\rangle\}_{k\in\mathcal{K}}$ maps $\mathcal{H}$ into ℓ^2.

Definition 1.8: Frame

A Bessel sequence $\{g_k\}_{k\in\mathcal{K}}, g_k \in \mathcal{H}$, is a **frame** for $\mathcal{H}$ if there exist constants $0 < A \le B < \infty$ (frame bounds) such that:

$$A\|x\|^2 \le \sum_{k\in\mathcal{K}} |\langle x, g_k\rangle|^2 \le B\|x\|^2 \quad \text{for all } x \in \mathcal{H}$$

Equivalently: $A\|x\|^2 \le \|\mathbb{T}x\|^2 \le B\|x\|^2$.

Key properties of frames:

- **Completeness**: If $A > 0$ and $\langle x, g_k\rangle = 0$ for all k, then $x = 0$
- **Boundedness**: If $B < \infty$, then $\mathbb{T}$ is continuous
- **Lower bound → left-invertibility**: $A > 0$ ensures perfect reconstruction is possible

The adjoint operator: For a bounded linear operator $\mathbb{T} : \mathcal{H} \to \ell^2$, define the **adjoint** $\mathbb{T}^* : \ell^2 \to \mathcal{H}$ by:

$$\langle \mathbb{T}x, c\rangle = \langle x, \mathbb{T}^* c\rangle$$

For frame analysis, $\mathbb{T}^*$ acts as:

$$\mathbb{T}^*\{c_k\}_{k\in\mathcal{K}} = \sum_{k\in\mathcal{K}} c_k g_k$$

This generalizes the Hermitian transpose $\mathbf{T}^{\mathrm{H}}$ from finite to infinite dimensions.

Synthesis Operator $\mathbb{T}^*$: Adjoint vs. Dual Synthesis

The **synthesis operator** $\mathbb{T}^* : l^2 \to \mathcal{H}$ is the adjoint of the analysis operator: $\mathbb{T}^* c = \sum c_k g_k$.
Critical distinction: $\mathbb{T}^* \neq \tilde{\mathbb{T}}^*$ (dual synthesis):

- $\mathbb{T}^*$: Adjoint only. $\mathbb{T}^*\mathbb{T}x = \mathbb{S}x \neq x$ (NO perfect reconstruction).
- $\tilde{\mathbb{T}}^* = \mathbb{S}^{-1}\mathbb{T}^*$: Gives $\tilde{\mathbb{T}}^*\mathbb{T}x = x$ (perfect reconstruction).

Why? In redundant frames, $\mathbb{T}^*\mathbb{T} = \mathbb{S} \neq I$. Reconstruction requires:

$$x = \mathbb{T}^*\mathbb{S}^{-1}\mathbb{T}x \quad \text{or equivalently} \quad x = \tilde{\mathbb{T}}^*\mathbb{T}x$$

1.3.1 The Frame Operator

Definition 1.14: Frame Operator

For a frame $\{g_k\}_{k\in\mathcal{K}}$ with analysis operator $\mathbb{T}$, the **frame operator** is:

$$\mathbb{S} = \mathbb{T}^*\mathbb{T} : \mathcal{H} \to \mathcal{H}$$

Acting on $x \in \mathcal{H}$:

$$\mathbb{S}x = \sum_{k\in\mathcal{K}} \langle x, g_k\rangle g_k$$

Why is $\mathbb{S}$ central?

- The frame condition $A\|x\|^2 \le \langle \mathbb{S}x, x\rangle \le B\|x\|^2$ directly characterizes frames via $\mathbb{S}$
- Perfect reconstruction: $x = \mathbb{T}^*\mathbb{S}^{-1}\mathbb{T}x$ (requires $\mathbb{S}$ invertible)
- In finite dimensions: $\mathbb{S} = \mathbf{T}^{\mathrm{H}}\mathbf{T}$ (Gram matrix of frame elements)

Theorem 1.15: Properties of the Frame Operator

1. **Linear and bounded**: $\mathbb{S}$ is continuous (composition of bounded operators)
2. **Self-adjoint**: $\mathbb{S}^* = (\mathbb{T}^*\mathbb{T})^* = \mathbb{T}^*\mathbb{T} = \mathbb{S}$
3. **Positive definite**: $\langle \mathbb{S}x, x\rangle = \|\mathbb{T}x\|^2 \ge A\|x\|^2 > 0$ for all $x \neq 0$
4. **Has unique positive square root**: There exists a unique self-adjoint positive operator $\mathbb{S}^{1/2}$ with $\mathbb{S} = \mathbb{S}^{1/2}\mathbb{S}^{1/2}$

Consequence: $\mathbb{S}$ is invertible, and the inverse $\mathbb{S}^{-1}$ is also self-adjoint, positive, bounded.

Proof outline:

- (1) Composition of bounded operators
- (2) $(AB)^* = B^*A^*$ gives self-adjointness
- (3) Lower bound: $\langle \mathbb{S}x, x\rangle \ge A\|x\|^2 > 0$
- (4) Spectral theorem: positive operators have unique positive square roots

Theorem 1.17: Frame Bounds from Spectrum
The **optimal** frame bounds are the extremal spectral values of $\mathbb{S}$:

$$A = \lambda_{\min}(\mathbb{S}) \quad \text{and} \quad B = \lambda_{\max}(\mathbb{S})$$

Intuition: Frame bounds characterize how $\mathbb{S}$ stretches/shrin vectors. Spectral bounds are tight (optimal).

1.3.2 The Canonical Dual Frame

Theorem 1.19: Canonical Dual Frame

Let $\{g_k\}_{k\in\mathcal{K}}$ be a frame with frame operator $\mathbb{S} = \mathbb{T}^*\mathbb{T}$ and bounds A, B. The **canonical dual frame** is:

$$\tilde{g}_k = \mathbb{S}^{-1}g_k, \quad k \in \mathcal{K}$$

This is also a frame for $\mathcal{H}$ with bounds $\tilde{A} = 1/B$ and $\tilde{B} = 1/A$.

Why is the canonical dual frame important?

- **Perfect reconstruction**: $x = \sum_{k\in\mathcal{K}} \langle x, g_k\rangle \tilde{g}_k$ for all $x \in \mathcal{H}$
- **Uniqueness**: The dual frame is the unique frame for perfect reconstruction

- **Reciprocal duality**: If $\{\tilde{g}_k\}$ is the dual of $\{g_k\}$, then $\{g_k\}$ is the dual of $\{\tilde{g}_k\}$

Key relationship: The analysis operator of the dual frame is:

$$\tilde{\mathbb{T}} = \mathbb{T}\mathbb{S}^{-1} = \mathbb{T}(\mathbb{T}^*\mathbb{T})^{-1}$$

and its frame operator is $\tilde{\mathbb{S}} = \mathbb{S}^{-1}$.

Proof outline:

- Invertibility: $A > 0$ ensures $\mathbb{S}$ is invertible by spectral theorem
- Dual bounds: Spectral analysis gives $\tilde{A} = 1/B$, $\tilde{B} = 1/A$
- Analysis operator: $\tilde{\mathbb{T}}x = \mathbb{T}(\mathbb{S}^{-1}x)$ (change of variables using self-adjointness)

1.3.3 Signal Expansions (Reconstruction)

Theorem 1.22: Frame Reconstruction

Let $\{g_k\}$ and $\{\tilde{g}_k\}$ be canonical dual frames. Every signal $x \in \mathcal{H}$ has two perfect reconstructions:

$$x = \sum_{k\in\mathcal{K}} \langle x, \tilde{g}_k\rangle g_k = \sum_{k\in\mathcal{K}} \langle x, g_k\rangle \tilde{g}_k$$

Equivalently: $\mathbb{T}^*\tilde{\mathbb{T}} = \tilde{\mathbb{T}}^*\mathbb{T} = \mathbb{I}_{\mathcal{H}}$.

In operator form:

$$x = \mathbb{T}^*\mathbb{S}^{-1}\mathbb{T}x = \mathbb{S}^{-1}\mathbb{S}x = x \quad \checkmark$$

Why two forms? The dual frame provides a natural way to recover x directly from the frame coefficients $\{\langle x, g_k\rangle\}$, without explicitly computing $\mathbb{S}^{-1}$. Both are mathematically equivalent.

1.3.4 Tight Frames

Definition 1.25: Tight Frame

A frame $\{g_k\}_{k\in\mathcal{K}}$ with **equal** tight bounds $A = B$ is called **tight**.

Theorem 1.26: Tight Frame Characterization

A frame is tight with bound A if and only if:

$$\mathbb{S} = A\mathbb{I}_{\mathcal{H}} \quad \Longleftrightarrow \quad x = \frac{1}{A}\sum_{k\in\mathcal{K}} \langle x, g_k\rangle g_k$$

This is **Parseval's identity** for tight frames. The dual is simple: $\tilde{g}_k = \frac{1}{A} g_k$.

Why tight frames matter:

- Perfect reconstruction without inverting $\mathbb{S}$ — just scale by $1/A$
- Example: Mercedes-Benz frame in $\mathbb{R}^2$: three unit vectors at 120 angles form tight frame with $A = 3/2$

1.3.5 Exact Frames

Definition 1.32: Exact Frame

A frame $\{g_k\}_{k\in\mathcal{K}}$ is **exact** if removing any single element leaves an incomplete set. Otherwise it is **inexact** (redundant).

Key equivalences:

- Exact frame = **Riesz basis** = frame with no redundancy
- Analysis operator $\mathbb{T}$ is **surjective** (onto ℓ^2)
- Representation in frame is **unique**: if $x = \sum c_k g_k = \sum a_k g_k$, then $c_k = a_k$ for all k (for exact frames, coefficients are $\langle x, \tilde{g}_k\rangle$ with minimal ℓ^2-norm)

Biorthonormality: Exact frames and their canonical duals satisfy:

$$\langle g_k, \tilde{g}_m\rangle = \delta_{k,m}$$

This is the infinite-dimensional generalization of biorthonormal bases.